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METHOD OF DESIGNING CASCADE BLADES WITH PRESCRIBED VELOCITY DISTRIBUTIONS IN COMPRESSIBLE POTENTIAL FLOWS

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SUMMARY

By use of the assumption that the pressure-volume relation is linear, a solution to the problem of designing a cascade for a given turning and with a prescribed velocity distribution along the blade in a potential flow of a compressible perfect fluid was obtained by a method of correspondence between potential flows of compressible and incompressible fluids. The designing of an isolated airfoil with a prescribed velocity distribution along the airfoil is considered as a special case of the cascade.

If the prescribed velocity distribution is not theoretically attainable, the method provides a means of modifying the distribution so as to obtain a physically significant blade shape. Numerical examples are included.

INTRODUCTION

In order to control boundary-layer growth, transition, and separation in the design of a cascade for a given turning, it is advantageous to prescribe the velocity along the blade as a function of the arc length along the blade and then to compute the blade shape. For an incompressible fluid, several solutions to this problem have been obtained. (See references 1 to 3.)

A similar solution for the two-dimensional potential flow of a compressible perfect fluid was developed at the NACA Lewis laboratory during 1948–49. This solution is based on the assumption that the pressure-volume relation is given by a linear approximation to the isentropic curve instead of the true curve. The flow pattern of the compressible fluid is obtained by a transformation from a corresponding flow of an incompressible fluid using the transformation developed by Lin (reference 4).

The method of solution consists in using the free-stream velocities upstream and downstream of the cascade and the prescribed dimensionless velocity distribution along the blade to select a suitable incompressible potential flow about the unit circle and then to determine the mapping function that transforms this incompressible flow into a compressible flow about a cascade of airfoils. The image of the unit circle under this mapping gives the cascade with the prescribed velocity distribution along the blade, provided the velocity distribution is theoretically attainable. If the velocity distribution is unattainable, methods are given for modifying the distribution so that a physically significant profile is obtained.

The problem of designing an isolated airfoil with a prescribed velocity distribution is considered as a special case of the cascade.

THEORY OF METHOD

CASCADE

In reference 4, Lin shows that if the pressure-density relation is (Symbols used in this report are defined in the appendix.)

$$p=C_1-\frac{C_2}{\rho} \tag{1}$$

then the compressible potential flow about a cascade of blades can be obtained by transforming the incompressible flow about the unit circle in the following manner: The complex potential function $F(\zeta)$ for the incompressible flow due to two complex sources at $\zeta=a_1$ and $\zeta=a_2$ outside the unit circle $|\zeta|=1$ is

$$F(\zeta)=A\log_e(\zeta-a_1)+\bar{A}\log_e\left(\zeta-\frac{1}{\bar{a}_1}\right)+B\log_e(\zeta-a_2)+ \bar{B}\log_e\left(\zeta-\frac{1}{\bar{a}_2}\right)+D \tag{2}$$

where A and B are complex constants with $Re\,A\geq0$ and $Re\,A=-Re\,B$, and D is an arbitrary complex constant. The bar indicates the complex conjugate. The mapping between the z-plane and the ζ-plane defined by

$$dz=g(\zeta)(\zeta-a_1)^{-1}(\zeta-a_2)^{-1}\,d\zeta-\frac{1}{4}\overline{[F'(\zeta)]^2[g(\zeta)]^{-1}(\zeta-a_1)(\zeta-a_2)\,d\zeta} \tag{3}$$

gives a compressible flow with the linear pressure-volume relation past a straight cascade of identical blades in the z-plane with the velocity potential ϕ_c and stream function ψ_c given by

$$\phi_c+i\psi_c=F(\zeta) \tag{4}$$

provided that $g(\zeta)$ is chosen to satisfy the following requirements:

$$\left.\begin{array}{l}\text{(a) The function } g(\zeta) \text{ is regular in closed region } R \text{ defined by } |\zeta| \geq 1. \\ \text{(b) The function } g(\zeta) \neq 0 \text{ in } R, \text{ except possibly at one point on the circle where } F'(\zeta)=0 \text{ (the order of the zero not to exceed 1).} \\ \text{(c) Along the circle } |\zeta|=1 \ \oint dz=0. \\ \text{(d) The function } g(\zeta) \text{ satisfies the inequality } |[F'(\zeta)][g(\zeta)]^{-1}(\zeta-a_1)(\zeta-a_2)|<2 \text{ in } R.\end{array}\right\} \tag{5}$$

The magnitude q and the direction α of the dimensionless velocity at any point in the z-plane are given by

$$\frac{2q}{1+\sqrt{1+q^2}}e^{-i\alpha}=\frac{F'(\zeta)(\zeta-a_1)(\zeta-a_2)}{g(\zeta)} \tag{6}$$

In order to use this transformation in designing a blade with a prescribed dimensionless velocity distribution along the blade in a cascade, the prescribed conditions are used to select a suitable incompressible flow about the unit circle and to determine the function $g(\zeta)$.

The prescribed conditions are the velocity distribution on the airfoil, the upstream velocity $q_1e^{i\alpha_1}$ and the downstream velocity $q_2e^{i\alpha_2}$. The upstream and downstream velocities are related by the isentropic-flow equations with $\gamma=-1$. This relation is

$$\frac{q_2^2 \cos^2\alpha_2}{1+q_2^2}=\frac{q_1^2 \cos^2\alpha_1}{1+q_1^2} \tag{7}$$

where the axis of the cascade has been taken along the y-axis for convenience and the flow is from left to right. (See fig. 1.)

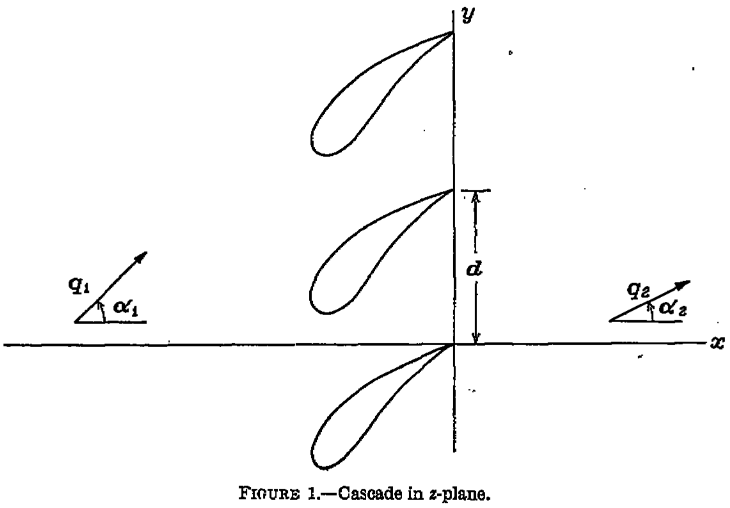

FIGURE 1.—Cascade in z-plane.

FLOW IN CIRCLE PLANE

The flow of an incompressible fluid about the unit circle is selected by determining the constants A, B, a_1, and a_2 in the complex potential

$$F(\zeta)=A \log_e (\zeta-a_1)+\overline{A} \log_e\left(\zeta-\frac{1}{\overline{a}_1}\right)+ B \log_e (\zeta-a_2)+\overline{B} \log_e\left(\zeta-\frac{1}{\overline{a}_2}\right)+D \tag{2}$$

from the given conditions. The constants A and B are obtained from the upstream and downstream velocities and the circulation and then a_1 and a_2 are determined by the range of the potential on the airfoil.

Circulation and cascade spacing.—The magnitude of the prescribed dimensionless velocity q along the airfoil is given as a function of the arc length $[q=q(s)]$ where the total arc length is taken to be 2π and is measured from the trailing edge along the lower surface. If $Q(s)$ is defined by

$$\begin{array}{ll} Q(s)=-q(s) & 0\leq s\leq s_n \\ Q(s)=q(s) & s_n\leq s\leq 2\pi \end{array} \tag{8}$$

where s_n is the leading-edge stagnation point, then

$$\phi_c(s)=\int_0^s Q(s)\,ds \tag{9}$$

$$\Gamma_c=\int_0^{2\pi} Q(s)\,ds \tag{10}$$

The circulation and the spacing of the cascade are related by

$$d=\frac{\Gamma_c}{q_1 \sin \alpha_1-q_2 \sin \alpha_2} \tag{11}$$

where d is the spacing. The quantities Γ_c, q_1, q_2, α_1, and α_2 are known so that the spacing is determined.

Determination of A.—The value of d from equation (11) is used to evaluate A and B because the spacing is also given by the absolute value of $\oint dz$ taken along a path around a_1 or a_2. (See fig. 2.) The axis of the cascade has been taken along the y-axis so that

$$id=\oint_{a_1} dz=-\oint_{a_2} dz \tag{12}$$

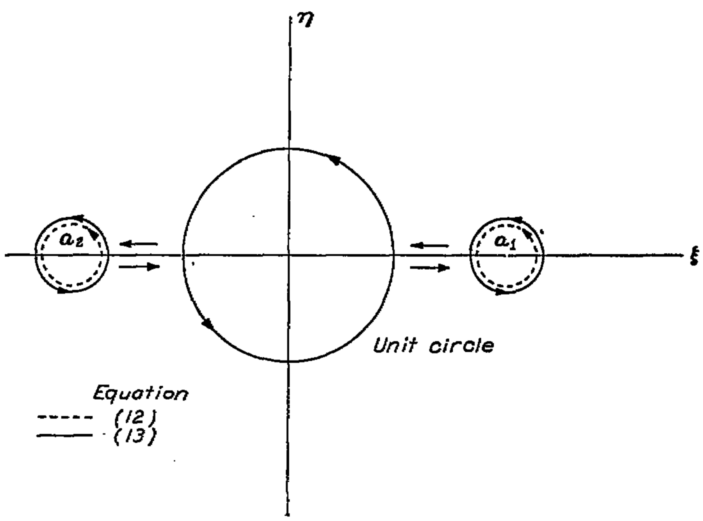

FIGURE 2.—Paths of integration in ζ-plane.

The second equality comes from the fact that the residues at infinity of

$$g(\zeta)(\zeta-a_1)^{-1}(\zeta-a_2)^{-1}$$

and

$$[F'(\zeta)]^2[g(\zeta)]^{-1}(\zeta-a_1)(\zeta-a_2)$$

in the expression for dz in equation (3) are zero and consequently

$$\oint_c dz+\oint_{a_1} dz+\oint_{a_2} dz=0 \tag{13}$$

where c is the unit circle. But by equation (5c),

$$\oint_c dz=0$$

so that

$$\oint_{a_1} dz=-\oint_{a_2} dz \tag{14}$$

The evaluation of equation (12) in terms of the potential $F(\zeta)$ is

$$id=\oint_{a_1}\frac{g(\zeta)}{(\zeta-a_1)(\zeta-a_2)}d\zeta-\frac{1}{4}\oint_{a_1}\overline{\frac{[F'(\zeta)]^2(\zeta-a_1)(\zeta-a_2)}{g(\zeta)}d\zeta} \tag{15}$$

But

$$F'(\zeta)=\frac{A}{\zeta-a_1}+\frac{\overline{A}}{\zeta-\frac{1}{\overline{a}_1}}+\frac{B}{\zeta-a_2}+\frac{\overline{B}}{\zeta-\frac{1}{\overline{a}_2}} \tag{16}$$

so that equation (15) reduces to

$$id=2\pi i\frac{g(a_1)}{(a_1-a_2)}-\frac{1}{4}\overline{\left[\frac{A^2(a_1-a_2)}{g(a_1)}\right]2\pi i} \tag{17}$$

$$=2\pi i\left[\frac{g(a_1)}{(a_1-a_2)}+\frac{\overline{A}^2}{4}\frac{\overline{(a_1-a_2)}}{\overline{g}(a_1)}\right] \tag{18}$$

At $\zeta=a_1$, equation (6) becomes

$$\frac{2q_1}{1+\sqrt{1+q_1^2}}e^{-i\alpha_1}=\frac{A(a_1-a_2)}{g(a_1)} \tag{19}$$

which on writing

$$K_1=\frac{2q_1}{1+\sqrt{1+q_1^2}}$$

reduces to

$$\frac{g(a_1)}{(a_1-a_2)}=\frac{A}{K_1}e^{i\alpha_1} \tag{20}$$

Substitution of the values from equation (20) in equation (18) gives

$$id=2\pi i\left[\left(\frac{A}{K_1}+\frac{\overline{A}K_1}{4}\right)e^{i\alpha_1}\right] \tag{21}$$

Hence, the bracketed expression in equation (21) must be a real number and

$$\frac{4A+\overline{A}K_1^2}{4K_1}\equiv re^{-i\alpha_1} \tag{22}$$

where

$$r^2=\left(\frac{4+K_1^2}{4K_1}Re\,A\right)^2+\left(\frac{4-K_1^2}{4K_1}Im\,A\right)^2 \tag{23}$$

From equation (22),

$$\frac{(4-K_1^2)Im\,A}{(4+K_1^2)Re\,A}=-\tan\alpha_1$$

or

$$Im\,A=-\frac{4+K_1^2}{4-K_1^2}Re\,A\tan\alpha_1 \tag{24}$$

$$Im\,A=-\sqrt{1+q_1^2}\,Re\,A\tan\alpha_1 \tag{25}$$

which gives the relation between $Re\ A$ and $Im\ A$.

Substitution of the value of $Im\ A$ from equation (24) in equation (23) yields

$$r^2=\left(\frac{4+K_1^2}{4K_1}\right)^2Re^2A+\left(\frac{4+K_1^2}{4K_1}\right)^2Re^2A\tan^2\alpha_1$$

$$=\left(\frac{4+K_1^2}{4K_1}\right)^2Re^2A\sec^2\alpha_1$$

or

$$r=\frac{4+K_1^2}{4K_1}Re\,A\,|\sec\alpha_1|$$

Hence

$$d=2\pi\left(\frac{4+K_1^2}{4K_1}Re\,A\,|\sec\alpha_1|\right) \tag{26}$$

Substitution of the value of d from equation (26) in equation (11) gives

$$2\pi\left(\frac{4+K_1^2}{4K_1}Re\,A\,|\sec\alpha_1|\right)=\frac{\Gamma_c}{q_1\sin\alpha_1-q_2\sin\alpha_2}$$

or

$$Re\,A=\frac{4K_1\Gamma_c\,|\cos\alpha_1|}{2\pi(4+K_1^2)(q_1\sin\alpha_1-q_2\sin\alpha_2)} \tag{27}$$

and $Re\ A$ is now determined. By use of this value of $Re\ A$ in equation (25), $Im\ A$ is obtained. Hence A is completely given by equations (27) and (25).

Determination of B.—From equation (12)

$$id=-\oint_{a_2}dz$$

$$=-\oint_{a_2}g(\zeta)(\zeta-a_1)^{-1}(\zeta-a_2)^{-1}d\zeta+$$

$$\frac{1}{4}\oint_{a_2}\overline{[F'(\zeta)]^2[g(\zeta)]^{-1}(\zeta-a_1)(\zeta-a_2)\,d\zeta}$$

$$id=-2\pi i\frac{g(a_2)}{(a_2-a_1)}+\frac{1}{4}\frac{\overline{B}^2\,\overline{(a_2-a_1)\,2\pi i}}{\overline{g}(a_2)}$$

$$=-2\pi i\frac{Be^{i\alpha_2}}{K_2}-\frac{2\pi i}{4}\overline{B}K_2e^{i\alpha_2}$$

$$=-2\pi i\left[\left(\frac{4B+K_2^2\overline{B}}{4K_2}\right)e^{i\alpha_2}\right]$$

where

$$K_2=\frac{2q_2}{1+\sqrt{1+q_2^2}}$$

The bracketed expression must be real, so that

$$\frac{(4-K_2^2)\,Im\,B}{(4+K_2^2)\,Re\,B}=-\tan\alpha_2 \tag{28}$$

But

$$Re\,B=-Re\,A \tag{29}$$

and equation (28) can be written

$$Im\,B=\frac{4+K_2^2}{4-K_2^2}Re\,A\tan\alpha_2$$

$$=\sqrt{1+q_2^2}\,Re\,A\tan\alpha_2 \tag{30}$$

Consequently, B is determined by equations (29) and (30) because $Re\,A$ is known from equation (27).

Determination of a_1 and a_2.—After A and B are known, the points a_1 and a_2 are to be selected to satisfy the single condition that the range of potential on the circle must equal the range of potential on the airfoil, that is,

$$\phi_c(2\pi)-\phi_c(s_n)=F[e^{i(\theta_t+2\pi)}]-F(e^{i\theta_n}) \tag{31}$$

where θ_t and θ_n are the trailing-edge and leading-edge stagnation angles, respectively. This condition is only one equation in two unknowns, a_1 and a_2; consequently, the values of a_1 and a_2 are not uniquely determined. By imposing an additional restriction that a_1 and a_2 are real and

$$a_1=-a_2 \tag{32}$$

unique values are obtained for a_1 and a_2 in all cases. In particular problems, however, some other restriction may be more useful, such as assigning a definite value for a_1 and computing a_2.

With the restriction given in equation (32), it is possible to express θ_t and θ_n in terms of a_1 and substitute these values in equation (31), but the resulting equation cannot be solved explicitly for a_1.

One method for obtaining a_1 is as follows: Let

$$\left.\begin{aligned}a_1&=e^k\\a_2&=-e^k\end{aligned}\right\} \tag{33}$$

where $k>0$. Then equation (2) becomes

$$F(\zeta)=A\log_e(\zeta-e^k)+\bar{A}\log_e(\zeta-e^{-k})+B\log_e(\zeta+e^k)+\bar{B}\log_e(\zeta+e^{-k})+D$$

or, for points on the unit circle $\zeta=e^{i\theta}$

$$F(e^{i\theta})=\phi_t(\theta)+i\psi_t(\theta)=Re\,A\log_e\frac{(e^{i\theta}-e^k)(e^{i\theta}-e^{-k})}{(e^{i\theta}+e^k)(e^{i\theta}+e^{-k})}+$$

$$iIm\,A\log_e\frac{e^{i\theta}-e^k}{e^{i\theta}-e^{-k}}+iIm\,B\log_e\frac{e^{i\theta}+e^k}{e^{i\theta}+e^{-k}}+D \tag{34}$$

Hence, $\phi(\theta)$ may be written in the form

$$\phi(\theta)=-2Re A\tanh^{-1}\frac{\cos\theta}{\cosh k}+(Im\,A+Im\,B)\tan^{-1}\frac{\tan\theta}{\tanh k}+$$

$$(Im\,A-Im\,B)\tan^{-1}\frac{\sin\theta}{\sinh k}+2Re A\tanh^{-1}\frac{\cos\theta_t}{\cosh k}-$$

$$(Im\,A+Im\,B)\tan^{-1}\frac{\tan\theta_t}{\tanh k}-(Im\,A-Im\,B)\tan^{-1}\frac{\sin\theta_t}{\sinh k} \tag{35}$$

where D has been chosen to make $\phi_t(\theta_t)=0$ and the angle convention is

$$-\frac{\pi}{2}<\tan^{-1}\frac{\sin\theta}{\sinh k}<\frac{\pi}{2}$$

and $\tan^{-1}\dfrac{\tan\theta}{\tanh k}$ is taken in the same quadrant and same direction as θ.

The velocity on the circle $v(\theta)$ is

$$v(\theta)=\frac{2Re A\sin\theta\cosh k}{\cosh^2 k-\cos^2\theta}+\frac{(Im\,A+Im\,B)\tanh k\sec^2\theta}{\tanh^2 k+\tan^2\theta}+$$

$$(Im\,A-Im\,B)\frac{\sinh k\cos\theta}{\sinh^2 k+\sin^2\theta}$$

$$=\frac{2}{\cosh 2k-\cos 2\theta}[2Re A\sin\theta\cosh k+$$

$$(Im\,A-Im\,B)\cos\theta\sinh k+(Im\,A+Im\,B)\sinh k\cosh k] \tag{36}$$

Equation (36) can be further simplified by defining λ as

$$\tan\lambda=\frac{(Im\,A-Im\,B)\sinh k}{2Re\,A\cosh k} \tag{37}$$

$$-\frac{\pi}{2}<\lambda<\frac{\pi}{2}$$

Then

$$v(\theta)=\frac{2\sqrt{4Re^2A\cosh^2 k+(Im\,A-Im\,B)^2\sinh^2 k}}{\cosh 2k-\cos 2\theta}\Bigg[\sin\theta\cos\lambda+$$

$$\cos\theta\sin\lambda+\frac{(Im\,A+Im\,B)\sinh k\cosh k}{\sqrt{4Re^2A\cosh^2 k+(Im\,A-Im\,B)^2\sinh^2 k}}\Bigg]$$

$$=\frac{4Re\,A\cosh k\sec\lambda}{\cosh 2k-\cos 2\theta}\left[\sin(\theta+\lambda)+\frac{(Im\,A+Im\,B)\sinh k}{2Re\,A\sec\lambda}\right] \tag{38}$$

The stagnation angles θ_t and θ_n are therefore the roots of the equation

$$\sin(\theta+\lambda)=\frac{-(Im\,A+Im\,B)\sinh k}{2Re\,A\sec\lambda} \tag{39}$$

The desired value of k is obtained as follows:

(1) Assume a value of k.
(2) Compute λ by equation (37).
(3) Obtain θ_t and θ_n from equation (39).
(4) Compute $\phi_t(\theta_t+2\pi)-\phi_t(\theta_n)$.
(5) Repeat (1) to (4) several times to obtain a plot of $\phi_t(\theta_t+2\pi)-\phi_t(\theta_n)$ as a function of k.

(6) Interpolate to obtain k such that

$$\phi_t(\theta_t+2\pi)-\phi_t(\theta_n)=\phi_c(2\pi)-\phi_c(s_n)$$

With k determined, the flow about the circle is known. The potential $\phi_t(\theta)$ and the velocity $v(\theta)$ for points on the circle are given by equations (35) and (38), respectively.

FUNCTION $g(\zeta)$

The function $g(\zeta)$ can be computed for points on the unit circle by using the prescribed velocity on the airfoil and the velocity on the unit circle to determine the real part of $g(\zeta)$. The imaginary part of $g(\zeta)$ can then be computed by Poisson's integral. Because of the restrictions imposed by the given conditions, however, $g(\zeta)$ is actually obtained in a slightly different manner, as shown in the following sections.

Airfoil with pointed trailing edge.—If an airfoil with a pointed trailing edge is desired, then $g(\zeta)$ must vanish at the trailing-edge stagnation point $\zeta=e^{i\theta_t}$. Hence, $g(\zeta)$ can be written in the form

$$g(\zeta)=\left(1-\frac{e^{i\theta_t}}{\zeta}\right)^n e^{f(\zeta)} \tag{40}$$

where $f(\zeta)$ is regular in the exterior of the unit circle and

$$n=1-\frac{\delta}{\pi} \tag{41}$$

where δ is the included trailing-edge angle of the airfoil. (See reference 4.)

Values of $g(\pm e^k)$.—Because the velocities are given for the compressible flow upstream and downstream of the cascade, the value of $g(\zeta)$ at $\zeta=\pm e^k$ is determined from equation (6),

$$\left.\begin{aligned} g(e^k)&=\frac{2e^kAe^{i\alpha_1}}{K_1}\\ g(-e^k)&=\frac{-2e^kBe^{i\alpha_2}}{K_2}\end{aligned}\right\} \tag{42}$$

In order that $g(\zeta)$ have these values, $f(\zeta)$ is written in the form

$$f(\zeta)=C(\zeta)+\frac{(e^{2k}-\zeta^2)(\zeta^2-e^{-2k})}{\zeta^2}H(\zeta) \tag{43}$$

where

$$C(\zeta)=\frac{1}{2}\left(1+\frac{e^k}{\zeta}\right)\log_e\left[\frac{2Ae^{i\alpha_1+k}}{K_1(1-e^{i\theta_t-k})^n}\right]+ \frac{1}{2}\left(1-\frac{e^k}{\zeta}\right)\log_e\left[\frac{-2Be^{i\alpha_2+k}}{K_2(1+e^{i\theta_t-k})^n}\right] \tag{44}$$

and $H(\zeta)$ is regular in the exterior of the unit circle with

$$\lim_{\zeta\to\infty}\zeta H(\zeta)=0 \tag{45}$$

The restriction on $H(\zeta)$ imposed by equation (45) is necessary so that $f(\zeta)$ (equation (43)) will be regular. By use of equation (43), $g(\zeta)$ is expressed as

$$g(\zeta)=\left(1-\frac{e^{i\theta_t}}{\zeta}\right)^n e^{C(\zeta)+\frac{(e^{2k}-\zeta^2)(\zeta^2-e^{-2k})}{\zeta^2}H(\zeta)} \tag{46}$$

and $g(\zeta)$ will be known when $H(\zeta)$ is determined. For the actual computation of the blade shape, only the values of $g(\zeta)$ on the unit circle are needed. Hence, it is only necessary to compute $H(\zeta)$ for points on the circle. If desired, the values of $H(\zeta)$ for any point in the exterior of the circle can be obtained from the values on the circle by Poisson's integral.

Determination of $Re\ H$ on circle.—By equation (4), the potentials $\phi_c(s)$ and $\phi_t(\theta)$ are equal at corresponding points. Thus, by matching these potentials a correspondence is established between points along the airfoil arc and the circle angles; that is, $s=s(\theta)$. By use of this correspondence, the magnitude of the prescribed velocity along the airfoil is obtained as a function of the circle angle $q=q(\theta)$. Hence, by taking absolute values of equation (6),

$$\frac{2q(\theta)}{1+\sqrt{1+q(\theta)^2}}=\frac{|F'(e^{i\theta})(e^{i\theta}-e^k)(e^{i\theta}+e^k)|}{|g(e^{i\theta})|} \tag{47}$$

for points on the circle. Substitution of the value of $g(\zeta)$ from equation (46) with $\zeta=e^{i\theta}$ and replacing $F'(e^{i\theta})$ by the velocity $v(\theta)$ (equation (38)) on the circle give

$$\frac{2q(\theta)}{1+\sqrt{1+q(\theta)^2}} =\frac{e^k|v(\theta)|(2\cosh 2k-2\cos 2\theta)^{\frac{1}{2}}}{[2-2\cos(\theta_t-\theta)]^{\frac{n}{2}}e^{[Re\ C(e^{i\theta})+(2\cosh 2k-2\cos 2\theta)Re\ H(e^{i\theta})]}}$$

or, with the equation solved for $Re\ H(e^{i\theta})$,

$$Re\,H(e^{i\theta})=\frac{\log_e\left\{\dfrac{|v(\theta)|(2\cosh 2k-2\cos 2\theta)^{\frac{1}{2}}}{K(\theta)[2-2\cos(\theta_t-\theta)]^{\frac{n}{2}}}\right\}-Re\ C(e^{i\theta})+k}{2\cosh 2k-2\cos 2\theta} \tag{48}$$

where

$$K(\theta)=\frac{2q(\theta)}{1+\sqrt{1+q(\theta)^2}} \tag{49}$$

Restrictions on $Re\ H(e^{i\theta})$.—Equation (45) imposes restrictions on the values of $Re\ H(e^{i\theta})$, as shown by writing $H(\zeta)$ in the form

$$H(\zeta)=h_0+\frac{h_1}{\zeta}+\frac{h_2}{\zeta^2}+\cdots \tag{50}$$

where h_0, h_1, h_2, . . . are complex constants.

For points on the circle, equation (50) becomes

$$\begin{aligned}H(e^{i\theta})&=Re\ H(e^{i\theta})+i\ Im\ H(e^{i\theta})\\ &=Re\ h_0+\sum_{j=1}(Re\ h_j\cos j\theta+Im\ h_j\sin j\theta)+\\ &\quad i\,[Im\ h_0+\sum_{j=1}(Im\ h_j\cos j\theta-Re\ h_j\sin j\theta)]\end{aligned} \tag{51}$$

Equation (51) is a Fourier expansion and

$$Re\ h_0=\frac{1}{2\pi}\int_0^{2\pi} Re\ H(e^{i\theta})\ d\theta \tag{52}$$

$$Re\ h_1=\frac{1}{\pi}\int_0^{2\pi} Re\ H(e^{i\theta}) \cos\theta\ d\theta \tag{53}$$

$$Im\ h_1=\frac{1}{\pi}\int_0^{2\pi} Re\ H(e^{i\theta}) \sin\theta\ d\theta \tag{54}$$

But equation (45) requires that

$$Re\ h_0=Im\ h_0=Re\ h_1=Im\ h_1=0 \tag{55}$$

Consequently, $Re\ H(e^{i\theta})$ must satisfy the equations

$$\int_0^{2\pi} Re\ H(e^{i\theta})\ d\theta=0 \tag{56}$$

$$\int_0^{2\pi} Re\ H(e^{i\theta}) \cos\theta\ d\theta=0 \tag{57}$$

$$\int_0^{2\pi} Re\ H(e^{i\theta}) \sin\theta\ d\theta=0 \tag{58}$$

Adjustment of $Re\ H(e^{i\theta})$.—If the values of $Re\ H(e^{i\theta})$ from equation (48) do not satisfy equations (56), (57), and (58), the values must be adjusted until the conditions are satisfied. One method for adjusting the function is to define $Re\ \tilde{H}(e^{i\theta})$ by

$$Re\ \tilde{H}(e^{i\theta})=Re\ H(e^{i\theta})-Re\ h_0-Re\ h_1 \cos\theta-Im\ h_1 \sin\theta \tag{59}$$

where $Re\ h_0$, $Re\ h_1$, and $Im\ h_1$ are given by equations (52), (53), and (54), respectively. The modified function $Re\ \tilde{H}(e^{i\theta})$ will then satisfy equations (56), (57), and (58). This method of modifying $Re\ H(e^{i\theta})$, however, changes the velocity distribution all along the profile and, if the correction terms in equation (59) are not small, these changes in the velocity may be extensive because

$$\frac{2\tilde{q}}{1+\sqrt{1+\tilde{q}^2}}=\frac{2q}{1+\sqrt{1+q^2}}e^{(Re\ h_0+Re\ h_1\cos\theta+Im\ h_1\sin\theta)\ (2\cosh 2k-2\cos 2\theta)}$$

where $\tilde{q}$ denotes the new velocity. In some cases, consequently, $Re\ H(e^{i\theta})$ can best be adjusted to satisfy the requirements by adding to $Re\ H(e^{i\theta})$ odd and even functions that have nonzero values only in small neighborhoods of the points $\theta=0$ and $\theta=-\pi$. The particular functions to be added to $Re\ H(e^{i\theta})$ and their range of values depend on the specific problem; no general method can be given for determining the functions.

Determination of $Im\ H(e^{i\theta})$.—After $Re\ H(e^{i\theta})$ satisfying equations (56), (57), and (58) is obtained, the function $Im\ H(e^{i\theta})$ is given by Poisson's integral (reference 5)

$$Im\ H(e^{i\theta})=\frac{1}{2\pi}\int_0^{2\pi} Re\ H(e^{i\tau}) \cot\left(\frac{\tau-\theta}{2}\right) d\tau \tag{60}$$

where the constant term in Poisson's integral has been taken as zero so that

$$Im\ h_0=\frac{1}{2\pi}\int_0^{2\pi} Im\ H(e^{i\theta})\ d\theta=0$$

as required by equation (55). Hence, $H(e^{i\theta})$ is determined for points on the unit circle by

$$H(e^{i\theta})=Re\ H(e^{i\theta})+i\ Im\ H(e^{i\theta})$$

Adjustment of $g(e^{i\theta})$.—By use of these values of $H(e^{i\theta})$ in equation (46), $g(\zeta)$ is determined for points on the circle,

$$g(e^{i\theta})=[1-e^{i(\theta_t-\theta)}]^n\ e^{[C(e^{i\theta})+(2\cosh 2k-2\cos 2\theta)\ H(e^{i\theta})]} \tag{61}$$

Because of the adjustments in $H(e^{i\theta})$, $g(e^{i\theta})$ may no longer satisfy condition (5d). If $g(e^{i\theta})$ does not satisfy the inequality for points on the circle, then the values of $g(\zeta)$ can be adjusted to satisfy the inequality by changing the second or higher harmonic terms in $H(\zeta)$ or by other methods. If the prescribed conditions are theoretically attainable, however, then no modification is necessary, not even in $Re\ H(e^{i\theta})$.

BLADE COORDINATES

By use of the values of $g(e^{i\theta})$ that satisfy all conditions, the blade coordinates are obtained from integration of equation (3), that is,

$$z=\int g(\zeta)(\zeta-e^k)^{-1}(\zeta+e^k)^{-1}d\zeta-\frac{1}{4}\int\frac{\overline{F'(\zeta)^2(\zeta-e^k)(\zeta+e^k)}}{g(\zeta)}d\zeta$$

which, on replacing $F'(e^{i\theta})$ by $v(\theta)e^{-i\left(\theta+\frac{\pi}{2}\right)}$ and writing

$$g(e^{i\theta})=g_1(\theta)e^{ig_2(\theta)}$$

reduces to

$$z=\int\left[g_1(\theta)(e^{2i\theta}-e^{2k})^{-1}-\frac{v(\theta)^2}{4g_1(\theta)}\overline{(e^{2i\theta}-e^{2k})}\right]e^{i\left[\theta+\frac{\pi}{2}+g_2(\theta)\right]}d\theta \tag{62}$$

COMPUTATIONAL PROCEDURE FOR CASCADE

An outline of the procedure for computing the blade shape is as follows:

(1) Obtain $\phi_c(s)$ and Γ_c from equations (9) and (10), respectively.

(2) Compute $Re\ A$, $Im\ A$, and $Im\ B$ by equations (27), (25), and (30), respectively.

(3) Obtain k as outlined in the text. Compute $\phi_t(\theta)$ and $v(\theta)$ by equations (35) and (38), respectively.

(4) Plot $\phi_c(s)$ and $\phi_t(\theta)$. By comparing the abscissas for equal values of these potentials, obtain s as a function of θ, which permits writing the prescribed velocity q as a function of θ, $q=q(\theta)$.

(5) Compute $Re\, H(e^{i\theta})$ by equation (48) and determine $Re\, h_0$, $Re\, h_1$, and $Im\, h_1$ by equations (52), (53), and (54), respectively. If these values are not zero, then adjust $Re\, H(e^{i\theta})$ either by equation (59) or by addition of functions so that $Re\, H(e^{i\theta})$ satisfies equations (56), (57), and (58).

(6) Obtain $Im\, H(e^{i\theta})$ by equation (60), using the adjusted values of $Re\, H(e^{i\theta})$.

(7) Obtain $g(e^{i\theta})$ by equation (61). The function $g(e^{i\theta})$ must satisfy inequality (5d) for points on the circle. If $g(e^{i\theta})$ does not satisfy the inequality, adjust $g(e^{i\theta})$ as suggested.

(8) After $g(e^{i\theta})$ has been adjusted to satisfy all conditions, the blade shape is obtained by integrating equation (62).

ISOLATED AIRFOIL

The problem of designing an isolated airfoil with a prescribed velocity distribution along the airfoil in a compressible potential flow with a given free-stream velocity can be considered as a degenerate case of the foregoing cascade problem in which the upstream and downstream velocities are equal and the spacing of the cascade becomes infinite. In this case, the singular points $\zeta=a_1$ and $\zeta=a_2$ move to infinity and the complex potential $F(\zeta)$ for the incompressible flow about the unit circle (equation (2)) becomes

$$F(\zeta)=-V\left(\zeta+\frac{1}{\zeta}\right)+\frac{\Gamma_i}{2\pi i}\log\zeta+D \tag{63}$$

where V is the incompressible free-stream velocity and is in the direction of the negative ξ-axis. The mapping function (equation (3)) becomes (reference 6)

$$dz=g(\zeta)\,d\zeta-\frac{1}{4}\overline{[F'(\zeta)]^2[g(\zeta)]^{-1}\,d\zeta} \tag{64}$$

where $g(\zeta)$ satisfies the following requirements:

$$\left.\begin{array}{l}\text{(a) The function } g(\zeta) \text{ is regular in the closed region } R \text{ defined by } |\zeta|\geq 1.\\ \text{(b) The function } g(\zeta)\neq 0 \text{ in } R\text{, except possibly at one point on the circle where } F'(\zeta)=0 \text{ (the order of the zero not to exceed 1).}\\ \text{(c) Along the circle } |\zeta|=1,\ \oint dz=0.\\ \text{(d) The function } g(\zeta) \text{ satisfies the inequality}\\ \qquad |[F'(\zeta)]\,[g(\zeta)]^{-1}|<2 \text{ in } R\end{array}\right\} \tag{65}$$

The velocity potential ϕ_c and the stream function ψ_c in the z-plane are given by

$$\phi_c+i\psi_c=F(\zeta) \tag{66}$$

The magnitude q and the direction α of the dimensionless velocity in the z-plane are given by

$$\frac{2q}{1+\sqrt{1+q^2}}\,e^{-i\alpha}=\frac{F'(\zeta)}{g(\zeta)} \tag{67}$$

In order to use the preceding transformation, the prescribed velocity distribution along the airfoil $q=q(s)$ and the free-stream velocity $q_1e^{i\alpha_1}$ are used to select the flow about the unit circle and to determine the function $g(\zeta)$. For the actual computation of the airfoil, only the values of the functions for points on the unit circle are needed.

FLOW IN CIRCLE PLANE

For points on the circle $\zeta=e^{i\theta}$, the complex potential (equation (63)) reduces to

$$\phi_i(\theta)=-2V(\cos\theta-\cos\theta_t)+\frac{\Gamma_i}{2\pi}(\theta-\theta_t) \tag{68}$$

when D is so chosen that $\phi_i=0$ at the trailing-edge stagnation point $\zeta=e^{i\theta_t}$ (fig. 3). The velocity is

$$v(\theta)=-ie^{-i\theta}\left(2V\sin\theta+\frac{\Gamma_i}{2\pi}\right) \tag{69}$$

Both $\phi_i(\theta)$ and $v(\theta)$ are completely determined when Γ_i, V, and θ_t are known. These quantities are obtained from the

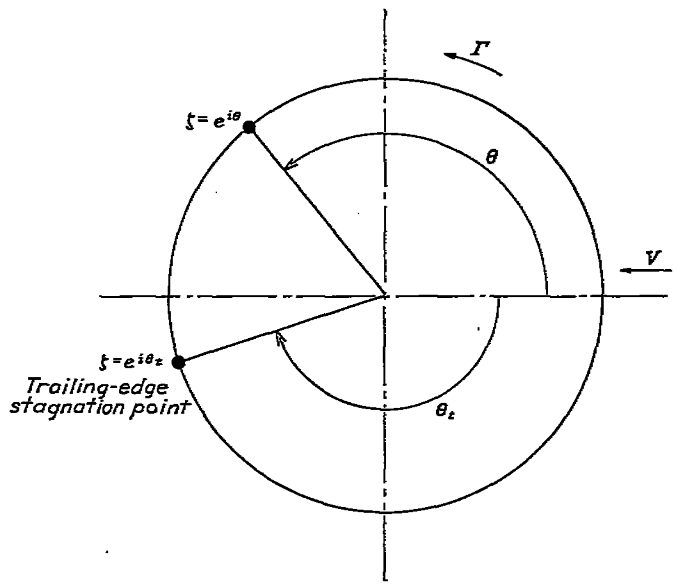

FIGURE 3.—Variables in circle plane.

prescribed velocity distribution by determining $\phi_c(s_n)$ and Γ_c from equations (9) and (10), respectively. By equation (66), the potentials are equal at corresponding points so that

$$\left.\begin{array}{c}\phi_{i,\,min}=\phi_{c,\,min}=\phi_c(s_n)\\ \Gamma_i=\Gamma_c\end{array}\right\} \tag{70}$$

Then θ_t is given by (reference 2)

$$\cot\theta_t+\theta_t=-\frac{\pi}{2}-\frac{\pi\phi_{i,\,min}}{\Gamma_i}=-\frac{\pi}{2}-\frac{\pi\phi_c(s_n)}{\Gamma_c} \tag{71}$$

and V is given by

$$V=-\frac{\Gamma_i}{4\pi\sin\theta_t}=-\frac{\Gamma_c}{4\pi\sin\theta_t} \tag{72}$$

The flow about the circle is obtained by using these values of Γ_t, V, and θ_t in equations (68) and (69).

FUNCTION $g(\zeta)$

As in the cascade case, if an airfoil with a pointed tail is desired, $g(\zeta)$ must vanish at the trailing-edge stagnation point $\zeta=e^{i\theta_t}$. Hence, $g(\zeta)$ can be written in the form

$$g(\zeta)=\left(1-\frac{e^{i\theta_t}}{\zeta}\right)^n e^{f(\zeta)} \tag{73}$$

where $f(\zeta)$ is regular in the exterior of the unit circle and

$$n=1-\frac{\delta}{\pi} \tag{41}$$

By using the prescribed velocity distribution on the airfoil and the circle velocity, the real part of $f(\zeta)$ can be computed for points on the circle and the imaginary part of $f(\zeta)$ can then be obtained by Poisson's integral. Hence $g(\zeta)$ would be determined. Because of the restrictions on $f(\zeta)$, this computational procedure is modified as shown in the next sections.

Restrictions on $f(\zeta)$.—Because the free-stream velocity is given for the compressible flow, the value of $g(\zeta)$ at infinity is determined by equation (67)

$$\lim_{\zeta\rightarrow\infty} g(\zeta)=\frac{Ve^{i(\alpha_1+\pi)}}{K_1} \tag{74}$$

where

$$K_1=\frac{2q_1}{1+\sqrt{1+q_1^2}}$$

By writing

$$f(\zeta)=b_0+\frac{b_1}{\zeta}+\frac{b_2}{\zeta^2}+\ .\ .\ .$$

where b_0, b_1, b_2, . . . are complex constants, then

$$\lim_{\zeta\rightarrow\infty} g(\zeta)=e^{b_0}$$

and

$$\left.\begin{aligned} Re\ (b_0)&=\log_e\frac{V}{K_1}\\ Im\ (b_0)&=\alpha_1+\pi\end{aligned}\right\} \tag{75}$$

Hence, the desired free-stream velocity will be obtained if b_0 satisfies equation (75).

Further restrictions are imposed on $f(\zeta)$ by the requirement

$$\oint g(\zeta)\ d\zeta-\frac{1}{4}\oint\overline{[F'(\zeta)]^2[g(\zeta)]^{-1}d\zeta}=0 \tag{76}$$

When the residues of these functions at infinity are considered, this requirement can be expressed in a more useful form. The function $g(\zeta)$ may be written as

$$g(\zeta)=e^{b_0}\left[1+\frac{b_1-ne^{i\theta_t}}{\zeta}+\text{terms in }\frac{1}{\zeta^j},\ j\geq 2\right]$$

and then

$$\oint g(\zeta)\ d\zeta=2\pi i e^{b_0}(b_1-ne^{i\theta_t}) \tag{77}$$

From the incompressible flow about the circle,

$$F'(\zeta)=-V+\frac{2Vi\sin\theta_t}{\zeta}+\frac{V}{\zeta^2}$$

and

$$[F'(\zeta)]^2[g(\zeta)]^{-1}=e^{-b_0}\left(V^2-\frac{4V^2i\sin\theta_t-nV^2e^{i\theta_t}+b_1V^2}{\zeta}+\text{terms in }\frac{1}{\zeta^j},\ j\geq 2\right)$$

therefore

$$\oint\overline{[F'(\zeta)]^2[g(\zeta)]^{-1}d\zeta}=-\overline{2\pi i e^{-b_0}(4V^2i\sin\theta_t-nV^2e^{i\theta_t}+b_1V^2)} \tag{78}$$

Then, by use of equations (77) and (78), equation (76) becomes

$$2\pi i e^{b_0}\ (b_1-ne^{i\theta_t})+\frac{1}{4}\ \overline{2\pi i e^{-b_0}V^2\ (4i\sin\theta_t-ne^{i\theta_t}+b_1)}=0$$

or

$$2\pi i e^{-\bar{b}_0}\ [e^{2\,Re\,b_0}\ (b_1-ne^{i\theta_t})-\frac{V^2}{4}\ (-4i\sin\theta_t-ne^{-i\theta_t}+\bar{b}_1)]=0 \tag{79}$$

Equating the real parts of equation (79) gives

$$\left(e^{2\,Re\,b_0}-\frac{V^2}{4}\right)(Re\ b_1-n\cos\theta_t)=0$$

But $e^{2\,Re\,b_0}-\dfrac{V^2}{4}$ cannot be zero except for zero free-stream velocity, hence

$$Re\ b_1=n\cos\theta_t \tag{80}$$

The imaginary parts of equation (79) give

$$e^{2\,Re\,b_0}\ (Im\ b_1-n\sin\theta_t)+\frac{V^2}{4}\ (Im\ b_1-n\sin\theta_t)+V^2\sin\theta_t=0$$

or

$$Im\ b_1=n\sin\theta_t-\frac{V^2\sin\theta_t}{e^{2\,Re\,b_0}+\dfrac{V^2}{4}}$$

But from equation (75)

$$e^{2\,Re\,b_0}=\frac{V^2}{K_1^2}$$

hence

$$Im\ b_1=\sin\theta_t\left(n-\frac{4K_1^2}{4+K_1^2}\right) \tag{81}$$

Consequently, equation (76) will be satisfied if $Re\, b_1$ and $Im\, b_1$ satisfy equations (80) and (81), respectively.

For convenience in computation, these restrictions on the values b_0 and b_1 are transformed into restrictions on the values of $Re\, f(e^{i\theta})$. For points on the circle,

$$f(e^{i\theta})=Re\, f(e^{i\theta})+i Im\, f(e^{i\theta})=b_0+\frac{b_1}{e^{i\theta}}+\frac{b_2}{e^{2i\theta}}+\ldots$$

$$=Re\, b_0+\sum_{j=1}(Re\ b_j \cos j\theta+Im\ b_j \sin j\theta)+$$

$$i\left[Im\ b_0+\sum_{j=1}(Im\ b_j \cos j\theta-Re\ b_j \sin j\theta)\right] \quad (82)$$

Equation (82) is a Fourier expansion and

$$Re\, b_0=\frac{1}{2\pi}\int_{\theta_t}^{\theta_t+2\pi} Re\, f(e^{i\theta})\, d\theta \quad (83)$$

$$Re\, b_1=\frac{1}{\pi}\int_{\theta_t}^{\theta_t+2\pi} Re\, f(e^{i\theta}) \cos \theta\, d\theta \quad (84)$$

$$Im\ b_1=\frac{1}{\pi}\int_{\theta_t}^{\theta_t+2\pi} Re\, f(e^{i\theta}) \sin \theta\, d\theta \quad (85)$$

Substitution of these values of $Re\ b_0$, $Re\ b_1$, and $Im\ b_1$ in equations (75), (80), and (81), respectively, gives

$$\frac{1}{2\pi}\int_{\theta_t}^{\theta_t+2\pi} Re\, f(e^{i\theta})\, d\theta=\log_e \frac{V}{K_1} \quad (86)$$

$$\frac{1}{\pi}\int_{\theta_t}^{\theta_t+2\pi} Re\, f(e^{i\theta}) \cos \theta\, d\theta=n \cos \theta_t \quad (87)$$

$$\frac{1}{\pi}\int_{\theta_t}^{\theta_t+2\pi} Re\, f(e^{i\theta}) \sin \theta\, d\theta=\sin \theta_t\left(n-\frac{4K_1^2}{4+K_1^2}\right) \quad (88)$$

Hence, $Re\, f(e^{i\theta})$ must satisfy equations (86) to (88).

Determination of $Re\, f(e^{i\theta})$.—By matching the potentials $\phi_c(s)$ and $\phi_t(\theta)$, a correspondence is established between points on the airfoil and the circle angles. The prescribed velocity distribution along the airfoil can therefore be written as a function of the circle angle $q=q(\theta)$. When absolute values of equation (67) are taken,

$$\frac{2q(\theta)}{1+\sqrt{1+q^2(\theta)}}=\frac{|F'(e^{i\theta})|}{|g(e^{i\theta})|} \quad (89)$$

for points on the circle. By replacing $F'(e^{i\theta})$ by the velocity $v(\theta)$ (equation (69)) and substituting $g(\zeta)$ from equation (73) with $\zeta=e^{i\theta}$, equation (89) becomes

$$\frac{2q(\theta)}{1+\sqrt{1+q^2(\theta)}}=\frac{\left|2V \sin \theta+\frac{\Gamma_i}{2\pi}\right|}{[2-2 \cos (\theta_t-\theta)]^{\frac{n}{2}} e^{Re\, f(e^{i\theta})}}$$

or, with the equation solved for $Re\, f(e^{i\theta})$,

$$Re\, f(e^{i\theta})=\log_e \frac{\left|2V \sin \theta+\frac{\Gamma_i}{2}\right|}{[2-2 \cos (\theta_t-\theta)]^{\frac{n}{2}} K(\theta)} \quad (90)$$

where

$$K(\theta)=\frac{2q(\theta)}{1+\sqrt{1+q^2(\theta)}}$$

The values of $Re\, f(e^{i\theta})$, as given by equation (90), must satisfy equations (86) to (88). If these equations are not satisfied, the values of $Re\, f(e^{i\theta})$ must be adjusted until the equations are satisfied. One method for adjusting the function is to define

$$Re\, \tilde{f}(e^{i\theta})=Re\, f(e^{i\theta})-\left(Re\, b_0-\log_e \frac{V}{K_1}\right)-(Re\, b_1-n \cos \theta_t) \cos \theta-$$

$$\left[Im\ b_1-\sin \theta_t\left(n-\frac{4K_1^2}{4+K_1^2}\right)\right]\sin \theta \quad (91)$$

where $Re\ b_0$, $Re\ b_1$, and $Im\ b_1$ are given by equations (83), (84), and (85), respectively. The function $Re\, \tilde{f}(e^{i\theta})$ will satisfy equations (86) to (88). If the corrections are large, this method may result in large changes in the velocity. In general, the method used for adjusting $Re\, f(e^{i\theta})$ will depend on the specific problem.

Determination of $Im\, f(e^{i\theta})$.—When $Re\, f(e^{i\theta})$ satisfying equations (86) to (88) is obtained, $Im\ f(e^{i\theta})$ is computed by Poisson's integral

$$Im\ f(e^{i\theta})=\frac{1}{2\pi}\int_{\theta_t}^{\theta_t+2\pi} Re\, f(e^{i\tau}) \cot \frac{\tau-\theta}{2} d\tau+(\alpha_1+\pi) \quad (92)$$

where the constant term in the integral has been taken equal to $\alpha_1+\pi$, so that

$$Im\, b_0=\frac{1}{2\pi}\int_{\theta_t}^{\theta_t+2\pi} Im\, f(e^{i\theta})\, d\theta=\alpha_1+\pi$$

as required by equation (75). Hence $f(e^{i\theta})$ is now known for points on the circle:

$$f(e^{i\theta})=Re\, f(e^{i\theta})+i\, Im\, f(e^{i\theta})$$

Determination of $g(e^{i\theta})$.—By use of these values of $f(e^{i\theta})$ in equation (73), $g(\zeta)$ is determined for points on the circle:

$$g(e^{i\theta})=[1-e^{i(\theta_t-\theta)}]^n e^{f(e^{i\theta})} \quad (93)$$

Because of the changes made in $Re\, f(e^{i\theta})$ to satisfy equations (86) to (88), $g(e^{i\theta})$ may no longer satisfy condition (65d). If $g(e^{i\theta})$ does not satisfy the inequality, $g(\zeta)$ can be adjusted by changing the second or higher harmonic terms in $f(e^{i\theta})$, or by some similar method.

AIRFOIL COORDINATES

The function $g(\zeta)$ has been obtained to satisfy all requirements; hence, the airfoil coordinates in the z-plane are given by equation (64) on integrating around the unit circle. For convenience, let $g(e^{i\theta})$ be written as

$$g(e^{i\theta})=g_1(\theta)e^{ig_2(\theta)}$$

From equation (64) the airfoil coordinates are then

$$x=-\int \frac{4g_1{}^2-\left(2V\sin\theta+\frac{\Gamma_t}{2\pi}\right)^2}{4g_1}\sin(g_2+\theta)\,d\theta \qquad (94)$$

$$y=\int \frac{4g_1{}^2-\left(2V\sin\theta-\frac{\Gamma_t}{2\pi}\right)^2}{4g_1}\cos(g_2+\theta)\,d\theta \qquad (95)$$

COMPUTATIONAL PROCEDURE FOR ISOLATED AIRFOIL

An outline of the procedure for computing the airfoil follows:

1. Obtain $\phi_c(s)$, Γ_c, and $\phi_c(s_n)$ from equations (9) and (10). By use of these values in equations (71) and (72), obtain θ_t and V, respectively. Then calculate $\phi_t(\theta)$ and $v(\theta)$ from equations (68) and (69), respectively.
2. Plot $\phi_c(s)$ and $\phi_t(\theta)$. By comparing equal values of these potentials, obtain s as a function of θ, which permits writing the prescribed velocity q as a function of θ, $q=q(\theta)$.
3. Compute $Re\,f(e^{i\theta})$ by equation (90); this function must satisfy equations (86) to (88). If these equations are not satisfied, $Re\,f(e^{i\theta})$ is adjusted to satisfy them as suggested by equation (91) or by similar methods.
4. Obtain $Im\;f(e^{i\theta})$ from equation (92) and compute $g(e^{i\theta})$ by equation (93). These values of $g(e^{i\theta})$ should be checked in inequality (65d). If $g(e^{i\theta})$ does not satisfy the inequality, it is further modified as previously suggested.
5. By use in equations (94) and (95) of the values of $g(e^{i\theta})$ that satisfy all requirements, the airfoil coordinates are obtained by integration.

ILLUSTRATIVE EXAMPLES

Isolated airfoil.—For this example, the prescribed velocity $q=q(s)$ was obtained from the incompressible velocity distribution about a symmetrical Joukowski profile, as computed by Lipman Bers at Syracuse University in the form of the ratio of actual velocity to free-stream velocity, by taking the dimensionless free-stream velocity to be 0.538. The resulting distribution is shown in figure 4 with the final velocity after adjusting $Re\;f(e^{i\theta})$. The computed profile (fig. 5) is slightly thicker than the Joukowski profile toward the nose and has some reflex camber. The peaks in the velocity distribution about the computed profile are lower than those that would occur in a compressible flow about the Joukowski profile with the same free-stream velocity and angle of attack because: (1) the circulation was kept the same as for the incompressible flow, which resulted in the reflex camber; and (2) the thickening of the profile reduced the curvatures in the vicinity of the velocity peaks.

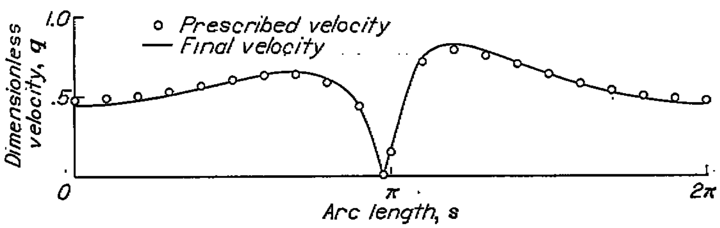

FIGURE 4.—Comparison of prescribed velocity and final velocity distributions for airfoil.

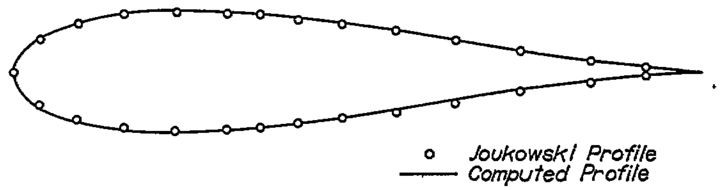

FIGURE 5.—Comparison of computed profile and Joukowski profile.

Cascade.—For this example, it was decided to design a cascade of blades having cusped tails and a velocity distribution along the blade like the distribution in the foregoing isolated-airfoil example. The free-stream conditions q_1, α_1, and α_2 were arbitrarily taken as

$$q_1=0.576$$

$$\alpha_1=10°$$

$$\alpha_2=0°$$

which gives

$$q_2=0.564$$

Adjustments to $ReH(e^{i\theta})$ to satisfy equations (86) to (88) modified the prescribed velocity somewhat and the final velocity is shown in figure 6. Figure 7 is the computed cascade.

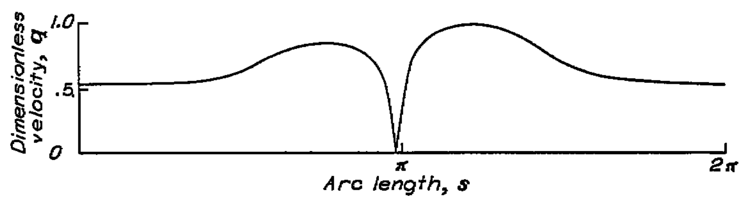

FIGURE 6.—Velocity distribution on cascade blade.

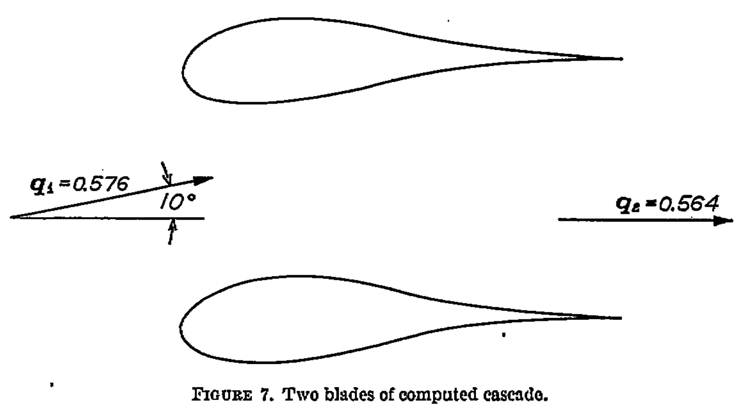

FIGURE 7. Two blades of computed cascade.

CONCLUDING REMARKS

The magnitude of the dimensionless velocity along the blade cannot be entirely prescribed arbitrarily as a function of the arc length, but is subject to some restrictions in addition to the conditions imposed on the regular functions $H(\zeta)$ and $f(\zeta)$. The magnitude must be finite everywhere along the profile and, by the method given, the velocity can be zero in, at most, two places—the leading-edge and the trailing-edge stagnation points. By a limiting process, however, the method can be extended to provide for additional stagnation points. The order of the zero of the dimensionless velocity at the trailing edge is determined by the included trailing-edge angle of the blade. Thus, for a cusp at the tail, the angle is zero and the dimensionless velocity need not be zero at the trailing edge.

If a velocity distribution is selected to satisfy these conditions, but otherwise is arbitrary, the resulting profile may not be a physically real blade but may result in a blade with zero or negative thickness in some portions of the blade. The zero or negative thickness is caused by specifying too-low velocities along parts of the blade; a physically real blade can be obtained by increasing the prescribed velocity along the blade.

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS,
LEWIS FLIGHT PROPULSION LABORATORY,
CLEVELAND, OHIO, *October 1, 1949.*

APPENDIX

SYMBOLS

The following symbols are used in this report:

A, B, C_1, C_2, D	complex constants
$a_1, a_2, \ldots$	location of complex sources in ζ-plane
$C(\zeta)$	function of ζ defined by equation (44)
d	spacing of cascade
$F(\zeta)$	complex potential function (incompressible flow)
$f(\zeta)$	regular function of ζ
$g(\zeta)$	regular function of ζ
$H(\zeta)$	regular function of ζ
$Re\ \tilde{H}(\zeta)$	function of ζ defined by equation (59)
Im	imaginary part
$K(\theta)$	function of θ defined by equation (49)
K_1	constant equal to $\frac{2q_1}{1+\sqrt{1+q_1^2}}$
K_2	constant equal to $\frac{2q_2}{1+\sqrt{1+q_2^2}}$
k	constant defined by equations (33)
n	number determined by included trailing-edge angle of blade
p	pressure
$Q(s)$	auxiliary function of s
q	magnitude of dimensionless velocity in compressible-flow plane (ratio of actual velocity to stagnation velocity of sound)
$q_1e^{i\alpha_1}$	dimensionless velocity upstream of cascade
$q_2e^{i\alpha_2}$	dimensionless velocity downstream of cascade
R	region in ζ-plane defined by $\lvert\zeta\rvert \geq 1$
Re	real part
r	number defined by equation (23)
s	arc length along blade
V	free-stream velocity (incompressible flow)
$v(\theta)$	velocity on unit circle (incompressible flow)
$z=x+iy$	complex variable (compressible-flow plane)
α	angle of velocity in compressible flow (measured from positive x-axis)
Γ	circulation
γ	ratio of specific heats
δ	included trailing-edge angle of blade
$\zeta=\xi+i\eta$	complex variable (incompressible-flow plane)
θ	circle angle (incompressible-flow plane)
λ	auxiliary variable defined by equation (37)
ρ	density
τ	variable of integration
ϕ	velocity potential
ψ	stream function

Subscripts:

c	compressible flow
i	incompressible flow
n	leading edge
t	trailing edge

Prime indicates a derivative.

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